ÇANKAYA UNIVERSITY
DEPARTMENT OF MATHEMATICS
December 16, 2004

Math 395
Metric Spaces
Midterm 2

Duration: 90 min.
Show all your work.

1. For each of the following statements, indicate whether the statement is true or false. If true give a proof. If false, give a counterexample.

a. (5 pts.) The set $\left\{(x,y) \mid x^2+y^2<1\right\}$ is a complete subset of $\mathbb{R}^2$.

b. (5 pts.) The set $\{(x,y) \mid x \leq y\}$ is a compact subset of $\mathbb{R}^2$.

c. (5 pts.) The function $f:(0,1) \longrightarrow \mathbb{R}$ defined by $f(x)=x$ is uniformly continuous.

d. (5 pts.) The set $\left\{(x,y) \mid x^2+y^2=2\right\}$ is not compact.

e. (5 pts.) Let X be a set with the discrete topology. Then X is compact if and only if X is finite.

2. (15 pts.) If X, Y and Z are topological spaces, and $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ are continuous funtions, then prove that $g \circ f: X \longrightarrow Z$ (defined by $(g \circ f)(x)=g(f(x))$ for all $x \in X$) is continuous.

3. (15 pts.) Show that the open interval (a,b) is homeomorphic to (c,d). [Hint: $f(x)=\frac{d-c}{b-a}(x-a)+c$.]

4. Let $X=(0,1]$ be equipped with the usual metric $d(x,y)=|x-y|$.

a. (6 pts.) Show that (X,d) is not complete.

b. (6 pts.) Let $\widetilde{d}(x,y)=\left|\frac{1}{x}-\frac{1}{y}\right|$ for $x,y \in X$. Show that $\widetilde{d}$ is a metric on X.

c. (6 pts.) Show that $\widetilde{d}$ is equivalent to d,

d. (6 pts.) Show that that $\left(X,\widetilde{d}\right)$ is complete.

5. (20 pts.) If A and B are subsets of a metric space (X,d) such that A is closed and B is compact, show that $A \cap B$ is compact.